INSTABILITY IN THE TENSION OF A CYLINDRICAL LAYER OF FLUID

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In the present study, we obtain an exact solution to nonsteady equations describing the symmetric rotational motion of an ideal fluid. The motion is interpreted as the motion of a cylindrical layer with free boundaries. Allowance is also made for the effect of surface tension in an analysis performed using a linear approximation. Asymptotes describing the growth of small perturbations are obtained, and numerical results are presented from calculation of the perturbations of the free boundaries of the layer as a function of the following parameters: Weber number; relative thickness of the layer; wavelength of the perturbation.

Basic Equation

We will examine the symmetric rotational motion of an ideal incompressible fluid in Lagrangian coordinates [1]:

$$r_{tt}-\frac{V}{r^3}+\frac{r}{\eta}(z_\zeta p_\eta - z_\eta p_\zeta)=0,\ z_{tt}-\frac{r}{\eta}(r_\zeta p_\eta - r_\eta p_\zeta)=0, \qquad (1)$$
$$r(r_\eta z_\zeta - r_\zeta z_\eta)=\eta.$$

Here, $\eta = r(\eta, \zeta, 0)$; $\zeta = z(\eta, \zeta, 0)$; t is time; p is pressure; $V(\eta, \zeta)$ is the square of the initial distribution of the angular momentum of a fluid particle about the z axis; the function V is assigned. The density of the fluid is taken equal to unity.

The group properties of system (1) were studied in [2], where the authors derived special forms of classifying function $V(\eta, \zeta)$ in which the main group is expanded. Let $V = V(\eta)$. Then system (1) allows the two-parameter subgroup $\langle \partial_\zeta + \partial_z, t\partial_z \rangle$ [2]. It should be noted that the initial conditions $r = \eta$, $z = \zeta$, $t = 0$ are an invariant manifold relative to this subgroup. Since the variables t, η, r and p serve as invariant subgroups here, then for system (1) we can seek only partially invariant solutions [3] of rank 2 with a defect 1 of the form $r = r(\eta, t)$, $z = z(\eta, \zeta, t)$, $p = p(\eta, t)$. In this case, we find from (1) that

$$r=\left\{2\int_0^\eta \eta\,[1+a(\eta)t]^{-1}d\eta + c(t)\right\}^{1/2},\quad z=[1+a(\eta)t]\,\zeta + b(\eta)\,t, \qquad (2)$$
$$p=\int r_\eta\left(\frac{V}{r^3}-r_{tt}\right)d\eta+\varphi(t)$$

with arbitrary functions $c(t)$, $(c(0)=0)$, $a(\eta)$, $b(\eta)$, $\varphi(t)$. Solution (2) can be interpreted as the motion of an infinite liquid cylinder $[c(t) \equiv 0]$ or cylindrical layer with free boundaries that is rotated about the z axis and is extended in the direction of this axis.

We set $a(\eta) \equiv k = \mathrm{const}$, $b(\eta) \equiv 0$ in (2) and assume that $c(t) \neq 0$ [at $c(t) = 0$, we obtain the problem of the tension of a fluid cylinder [4]). Thus,

$$r = m(\eta, t)\eta,\ z = \tau\zeta,\ m = (1/\tau + c/\eta^2)^{1/2},\ \tau = 1 + kt. \qquad (3)$$

In this case, the cylindrical layer can be considered finite. In fact, let the region occupied by the fluid at the initial moment of time be represented as a cylindrical layer $\Omega = \{\eta, \zeta | \eta_1 < \eta < \eta_2,\ 0 < \zeta < h\}$. The planes $\zeta = 0$, $\zeta = h$ are impermeable walls while the cylindrical surfaces $\eta = \eta_{1,2}$ are free boundaries. The initial field of velocity on Ω has the form $w_0 = k\zeta$, $u_0 = [c'(0) - k\eta^2]/2\eta$, so that the constant k is determined by the initial velocity $W = kh$ of the solid wall $\zeta = h$.

To study the evolution of the free boundaries of the cylindrical layer, we designate the internal radius as $r_1(t)$ and the external radius as $r_2(t)$. We then use (3) to obtain $r_1(t) = m(\eta_1, t)\eta_1$, $r_2(t) = m(\eta_2, t)\eta_2$, which means that

$$r_2^2(t) - r_1^2(t) = (\eta_2^2 - \eta_1^2)/\tau. \qquad (4)$$

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Final relation (4) is the law of conservation of the volume of the cylindrical layer. As follows from (3)-(4), during motion the fluid keeps the form of a hollow cylinder. Meanwhile, the solid wall $\zeta = 0$ remains stationary. The top wall moves in accordance with the law $z = \tau h$. If $k > 0$, then the cylindrical layer is extended along the z axis. At $k < 0$, the moving plane encounters the stationary plane during the time $t = 1/|k|$. If u_{10} is the initial velocity of the internal surface, then $c'(0) = 2\eta_1 u_{10} + k\eta_1^2$. We will assume that $c'(0) = 0$, so that motion is determined only by the constant $k = W/h$ — the initial tension or compression along the z axis. At $k = 0$, $c'(0) \neq 0$, we obtain the plane flow of a ring. This problem was studied in [4, 5].

Let σ_1 and σ_2 represent the values of surface tension on the internal and external surfaces. We can use the dynamic condition on the boundaries $p(r_2(t), t) - p(r_1(t), t) = \sigma_1/r_1(t) + \sigma_2/r_2(t)$ and Eqs. (2), (4) to find a second-order ordinary differential equation for the function $c(t)$. Instead of $c(t)$, it is convenient to introduce the new function $g = 1 + \eta_1^{-2}\tau c$ and change over to the variable $\mu = (1 + kt)^2 = \tau^2$ in place of t. Then the above equation in $g(\mu)$ has the form

$$g'' \ln\left(1+\frac{\varepsilon}{g}\right)+\frac{g'^2}{2}\left(\frac{1}{g+\varepsilon}-\frac{1}{g}\right)+\frac{3\varepsilon}{8\mu^2}-$$
$$-\frac{1}{2k^2\eta_1^4}\int\limits_1^{1+\varepsilon}\frac{V(\eta_1\sqrt{y})}{(g-1+y)^2}\,dy+\frac{1}{\mu^{1/4}}\left(\frac{S_1}{\sqrt{g}}+\frac{S_2}{\sqrt{g+\varepsilon}}\right)=0,\quad g(1)=1,\quad g'(1)=0, \tag{5}$$

where $\varepsilon = (\eta_2/\eta_1)^2 - 1 > 0$; $S_j = \sigma_j/\eta_1^3 k^2$ $(j = 1, 2)$ represent the Weber numbers (as noted above, the density of the fluid is assumed to be equal to unity). Using the function $g(\mu)$, we determine the radii of the internal and external surfaces from the formulas

$$r_1 = \eta_1\mu^{-1/4}g^{1/2}, \quad r_2 = \eta_1\mu^{-1/4}(g+\varepsilon)^{1/2}. \tag{6}$$

In the case of potential motion of the layer, $V \equiv 0$, and it is not hard to see from (5) that $g(\mu) \leqslant 1$ for all $\mu \geqslant 1$. More accurate calculations show that the following inequality is valid

$$-\left[(1-g)\frac{3\varepsilon}{4}-4S_1\sqrt{g}-4S_2\sqrt{g+\varepsilon}+\gamma\right]^{1/2}\leqslant$$
$$\leqslant\left[\ln\left(1+\frac{\varepsilon}{g}\right)\right]^{1/2}\frac{dg}{d\mu}\leqslant-\left[\gamma-4S_1\sqrt{g}-4S_2\sqrt{g+\varepsilon}\right]^{1/2} \tag{7}$$

$(\gamma = 4S_1 + 4S_2\sqrt{1+\varepsilon})$. It is clear from this that there exists $\mu_* > 1$ such that $g(\mu_*) = 0$. Returning to Eqs. (6), we obtain $r_1(t_*) = 0$, $r_2(t_*) = \eta_1\mu_*^{-1/4}\varepsilon^{1/2}$ at the moment of time $t_* = (\sqrt{\mu_*} - 1)/k$, $k > 0$. Estimate (7) can be used to show that the velocity of the internal surface increases without limit. Meanwhile, $dr_1/dt \sim c_0 r_1^{-1}[-\ln(r_1\sqrt{\tau}/\eta_1)]^{-1/2}$ at $r_1 \to 0$, $c_0 = k\eta_1^2\mu_*^{1/2}(3\varepsilon/8)^{1/2}$. This reflects the fact that a hydraulic shock — collapse of the cavity — occurs at the moment of the cavity's disappearance.

Unfortunately, this motion cannot continue as the motion of a cylindrical jet during the time $t > t_*$. In fact, if $g = 0$ at $t \geqslant t_*$, then it is necessary to take $c = -\eta_1^2/\tau$. We then obtain

$$r=\left(\frac{\eta^2-\eta_1^2}{\tau}\right)^{1/2}, \quad z=\tau\zeta, \quad p=-\frac{3}{8\tau^3}\eta^2+\varphi(t). \tag{8}$$

It is clear that this solution describes the motion, at $t \geqslant t_*$, of a solid cylindrical jet with a free boundary $r_3(t) = [(\eta_2^2 - \eta_1^2)/\tau]^{1/2}$. In this case, $\varphi(t) = \sigma_2/r_3(t) + 3(\eta_2^2 - \eta_1^2)/8\tau^3$. It can be shown that the total energy of the cylindrical layer remains finite at the moment of collapse but does not coincide with the energy of the jet (8) at this moment. The difference is proportional to $\lim\limits_{t\to t_*}(g')^2\ln(1+\varepsilon/g)$ and is nontrivial by virtue of (7). In addition, motion in the cylindrical layer at $t \to t_*$ increases without limit and thus does not coincide with pressure in the jet distribution in accordance with law (8).

<u>Note 1.</u> Hydrodynamic equations (1) for symmetric rotational motion are invariant under the transformations $t' = t + a$, $\eta' = \sqrt{\eta^2 + b(\zeta)d}$, $\zeta' = \zeta$, $r' = r$, $z' = z$, $p' = p$ with arbitrary parameters a and d and the function $b(\zeta)$ [2]. Having set $a = -t_*$, $b = 1$, $d = -\eta_1^2$, we find that (8) reduces to the well-known solution obtained by Ovsyannikov [4] with a linear velocity field.

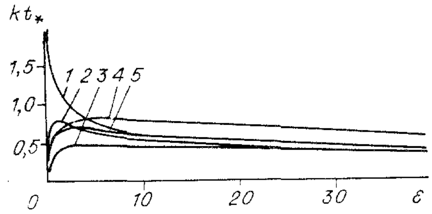

Fig. 1

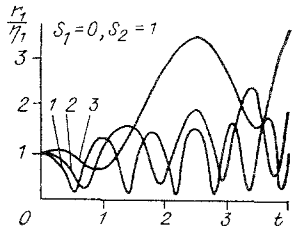

Fig. 2

Figure 1 shows numerically constructed relations for the dimensionless time of collapse $kt_*(\varepsilon)$ of the cavity. Curve 1 corresponds to $S_1 = S_2 = 0$, 2 to $S_1 = 0, 1$, $S_2 = 0$, 3 to $S_1 = S_2 = 1$, while the angular moment is equal to zero [the function $V(\eta)$ in Eq. (5) is equal to zero]. Curves 4 ($V = 0.2k^2\eta_1^4 y^2$, $S_1 = 1$, $S_2 = 0$) and 5 ($V = 0.2k^2\eta_1^4 y$, $S_1 = 1$, $S_2 = 0$) show that collapse of the cavity can also take place with zero rotation.

Oscillatory motion of the cylindrical layer can be established when the function $V(\eta)$ is assigned under certain conditions. Figure 2 shows graphs of the dependences of the radius of the internal free boundary on time t with $V = 0.2k^2\eta_1^4(1 + y^2)$, $S_1 = 0$, $S_2 = 1$ and $\varepsilon = 0.5$, 1, 3 (lines 1-3, respectively). An unbounded increase in internal radius is possible in those cases in which the inertial forces connected with rotation of the liquid about the z axis predominate over other forces. Such motion can be realized, for example, when $S_1 = 0$, $S_2 = 0$, $V = 0.2k^2\eta_1^4(1 + y^2)$.

Small Perturbation of a Cylindrical Layer

With allowance for capillary forces, the problem of the evolution of small perturbations of an arbitrary potential flow of an ideal incompressible fluid has the form [6]

$$\operatorname{div} M^{-1}M^{*-1}\nabla\Phi = 0,\ \xi \in \Omega,\ t \geqslant 0; \tag{9}$$

$$\Phi_t = \left[\frac{\partial p}{\partial n_{\Gamma_t}} + \sigma\left(\frac{1}{R_1^2} + \frac{1}{R_2^2}\right)\right] R + \sigma\bar{\Delta}_\Gamma(t) R, \quad \xi \in \Gamma, \quad t \geqslant 0; \tag{10}$$

$$R = \frac{|\nabla f|}{|M^{*-1}\nabla f|}\,\mathbf{n}\cdot\left(\mathbf{s} + \int_0^t M^{-1}M^{*-1}\nabla\Phi\,dt\right), \quad \xi \in \Gamma, \quad t \geqslant 0; \tag{11}$$

$$\Phi = 0,\ \operatorname{div}\mathbf{s} = 0,\ \xi \in \Omega,\ t = 0. \tag{12}$$

Here, M is the Jacobian matrix of the mapping $\xi \to \mathbf{x}(\xi, t)$ of the initial region Ω onto the flow region Ω_t at $t > 0$ with elements $M_{ij} = \partial x_i/\partial \xi_j$ (i, j = 1, 2, 3); M^* is the conjugate matrix; Γ is the boundary of Ω; $f(\xi) = 0$ is its equation; $\mathbf{n}(\xi)$ is a normal to Γ; Γ_t is the boundary of Ω_t; R_1 and R_2 are the principal radii of curvature of its normal sections; $\partial p/\partial n_{\Gamma_t}$ is the derivative of pressure with respect to the normal to Γ_t; $\bar{\Delta}_\Gamma(t)$ is the Laplace–Belgrami operator with the coefficients $E = |M\xi_\alpha|^2$, $G = |M\xi_\beta|^2$, $F = (M\xi_\alpha, M\xi_\beta)$ ($(\alpha, \beta) \to \xi(\alpha, \beta)$ is the regular parameterization of the boundary Γ); $\mathbf{s}(\xi)$ ($\xi \in \Gamma$) is the vector of displacement of points of the boundary characterizing the initial perturbation of the flow region. The function $R(\xi, t)$ ($\xi \in \Gamma$) is the deviation of the free boundary in perturbed motion from the free boundary in unperturbed motion. It most clearly characterizes the effect of small perturbations on motion with the free boundary.

For the potential motion of a cylindrical layer (3), the mapping $\xi \to \mathbf{x}(\xi, t) = (m(\eta, t)\xi_1, m(\eta, t)\xi_2, \tau\zeta)$ and the Jacobian matrix M permit the representation

$$M = mE_1 + \eta m_\eta Q + \tau E_2 \quad (\eta^2 = \xi_1^2 + \xi_2^2,\ \xi_3 = \zeta),$$

where $E_1 = \operatorname{diag}(1, 1, 0)$; $E_2 = \operatorname{diag}(0, 0, 1)$, while the elements of the matrix Q are equal $\xi_i\xi_j/\eta^2$ (i, j = 1, 2) and 0 at other values of i, j. Since $QE_2 = 0$, $Q^2 = Q$, $E_1E_2 = 0$, we obtain the following expression for the inverse matrix M^{-1}

$$M^{-1} = \frac{1}{m}E_1 - \tau\eta m_\eta Q + \frac{1}{\tau}E_2.$$

Considering that $M = M^*$, $Q\nabla\Phi = (\xi_1, \xi_2, 0)\Phi_\eta/\eta$, after completing certain transformations we can use (9) to obtain the following equation for the function $\Phi(\eta, \theta, \zeta, t)$

$$\Phi_{\eta\eta} + \frac{\eta^2 - \tau c}{(\eta^2 + \tau c)\,\eta}\Phi_\eta + \frac{\eta^2}{(\eta^2 + \tau c)^2}\Phi_{\theta\theta} + \frac{\eta^2}{(\eta^2 + \tau c)\,\tau^3}\Phi_{\zeta\zeta} = 0 \tag{13}$$

with $(\eta, \theta, \zeta) \in \Omega = \{\eta_1 < \eta < \eta_2,\ 0 \leqslant \theta \leqslant 2\pi,\ 0 < \zeta < h\}$.

In the transformation of boundary condition (11), it is necessary to take $\sigma = \sigma_1$ at $\eta = \eta_1$, $\sigma = \sigma_2$ at $\eta = \eta_2$. We introduce the notation $m_j(t) = m(\eta_j, t)$ $(j = 1, 2)$ since the parameterization of the free boundaries $\Gamma_j(\eta = \eta_j)$ is $\xi(\theta, \zeta) = (\eta_j \cos\theta, \eta_j \sin\theta, \zeta)$, then

$$E = (\eta_j m_j)^2,\quad G = \tau^2,\quad F = 0,\quad \overline{\Delta}_{\Gamma_j}(t)\, R = \frac{1}{(\eta_j m_j)^2}\frac{\partial^2 R}{\partial\theta^2} + \frac{1}{\tau^2}\frac{\partial^2 R}{\partial\zeta^2}.$$

Then the equation of the free boundaries Γ_j is $f = \xi_1^2 + \xi_2^2 - \eta_j^2 = 0$. This in turn means that $R_1^{-2} + R_2^{-2} = (\eta_j m_j)^{-2}$, $|\nabla f|/|M^{*-1}\nabla f| = 1/\tau m_j$, $\mathbf{n}_j = \mp(\xi_1, \xi_2, 0)/\eta_j$. Also, we find from (2)-(3) and (11) that

$$\frac{\partial p}{\partial n_{\Gamma_t}} = \pm\, \eta_j m_{jtt}\quad (\eta = \eta_j); \tag{14}$$

$$R = \frac{1}{\tau m_j}\left(s_j \mp \int_0^t \tau^2 m_j^2 \Phi_\eta\, dt\right)\quad (\eta = \eta_j),\quad s_j = \mathbf{s}\cdot\mathbf{n}_j\quad (j = 1, 2), \tag{15}$$

while boundary condition (11) leads to the relations

$$\Phi = \left[\pm\frac{\eta_j m_{jtt}}{\tau m_j} + \frac{\sigma_j}{\tau\eta_j^2 m_j^3}\left(\frac{\partial^2}{\partial\theta^2} + \frac{\eta_j^2 m_j^2}{\tau^2}\frac{\partial^2}{\partial\zeta^2} + 1\right)\right]\left(s_j \mp \int_0^t \tau^2 m_j^2 \Phi_\eta\, dt\right) = 0\quad (\eta = \eta_j,\ j = 1, 2), \tag{16}$$

where the top sign corresponds to j = 1 and the bottom sign to j = 2.

Since the perturbation of velocity is determined from the formula [6]

$$\mathbf{U} = M^{*-1}\nabla\Phi, \tag{17}$$

the condition of nonflow through the solid walls $\zeta = 0$, $\zeta = h$ is equivalent to

$$\Phi_\zeta = 0\ (\zeta = 0,\ \zeta = h). \tag{18}$$

Thus, to analyze the behavior of small perturbations of a cylindrical layer of fluid, it is necessary to find the function $\Phi(\eta, \theta, \zeta, t)$ as the solution of initial boundary-value problem (13), (16), (18), (12) and then calculate $R^j(t) = R|_{\eta=\eta_j}$ using Eq. (15).

It should be noted that in the case $\sigma_1 = \sigma_2 = 0$, in accordance with (14)-(15) $\partial p/\partial n_{\Gamma_t} < 0$ at $\eta = \eta_j$. Thus, problem (13), (16), (18), (12) is correctly formulated [7]. When $\sigma_1 > 0$, $\sigma_2 > 0$, it is correctly formulated in accordance with Adamar's results, regardless of the value of $\partial p/\partial n_{\Gamma_t}$ [6].

Construction of Perturbed Motion and Its Asymptotic Analysis

We note that the variables (η, τ), θ, ζ in problem (13), (16), (18) are separated. With allowance for Eqs. (18), we write

$$\Phi = \sum_{n=0}^{\infty}\sum_{\lambda=0}^{\infty} A_{n\lambda}(\eta, t)\, e^{i\lambda\theta}\cos\left(\frac{n\pi}{h}\zeta\right)\quad (i = \sqrt{-1}). \tag{19}$$

After inserting (19) into (13), we obtain an equation for $A(\eta, t) \equiv A_{n\lambda}(\eta, t)$:

$$A_{\eta\eta} + \frac{\eta^2 - \tau c}{\eta(\eta^2 + \tau c)} A_\eta - \left[\frac{\eta^2\lambda^2}{(\eta^2 + \tau c)^2} + \frac{\eta^2 q^2}{\tau^3(\eta^2 + \tau c)}\right] A = 0 \tag{20}$$

$(q = n\pi/h)$; it can be proven that its general solution has the form

$$A(\eta, t) = B_1(t)\, I_\lambda\left[\frac{q}{\tau^{3/2}}(\eta^2 + \tau c)^{1/2}\right] + B_2(t)\, K_\lambda\left[\frac{q}{\tau^{3/2}}(\eta^2 + \tau c)^{1/2}\right] \tag{21}$$

with arbitrary functions $B_1(t)$, $B_2(t)$ (I_λ, K_λ are modified Bessel functions of the first and second kinds).

Let

$$N_j = \left[a_j \mp \frac{1}{k}\int_1^\tau \tau^2 m_j^2 A_\eta\, dt\right]\Big/\eta_1\quad (\eta = \eta_j), \tag{22}$$

where $a_j \equiv a_{n\lambda,j}$ are coefficients of the Fourier series of the initial displacements of the surface $s_j(\theta, \zeta)$. We also introduce the new functions $A_j(\tau) = A(\eta_j, t)/k\eta_1^2$ and designate

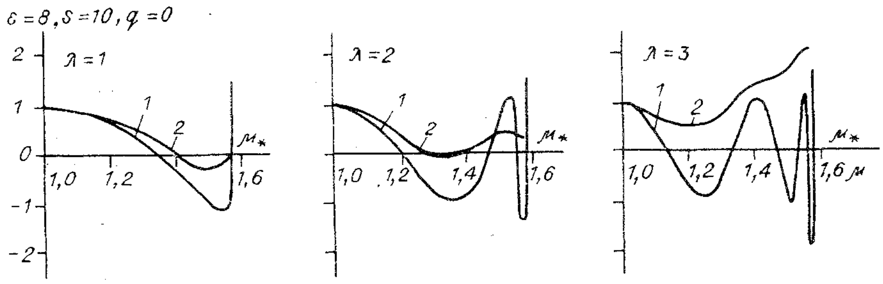

Fig. 3

$$f_j = I_\lambda\left[\frac{q}{\tau^{3/2}}(\eta_j^2+\tau c)^{1/2}\right],\quad g_j = K_\lambda\left[\frac{q}{\tau^{3/2}}(\eta_j^2+\tau c)^{1/2}\right].$$

With $q \neq 0$, we find from (21) that

$$A_\eta = \frac{k\eta_1^2}{\Delta}\left[\left(g_2\frac{dI_\lambda}{d\eta} - f_2\frac{dK_\lambda}{d\eta}\right)A_1 + \left(f_1\frac{dK_\lambda}{d\eta} - g_1\frac{dI_\lambda}{d\eta}\right)A_2\right],\quad \Delta = f_1g_2 - f_2g_1.$$

After inserting the resulting expressions into (16) and (22), we obtain a system of four first-order ordinary differential equations:

$$\frac{dA_j}{d\tau} = \left[\pm\frac{\eta_j m_{j\tau\tau}}{\eta_1\tau m_j} - \frac{\eta_1^2 S_j}{\eta_j^2\tau m_j^3}\left(\frac{m_j^2\beta_j^2}{\tau^2}+\lambda^2-1\right)\right]N_j; \tag{23}$$

$$\frac{dN_1}{d\tau} = \frac{\tau^2 m_1^2}{\Delta}\left(v_1A_1 + \frac{1}{g}A_2\right),\quad \frac{dN_2}{d\tau} = \frac{\tau^2 m_2^2}{\Delta}\left(\frac{\sqrt{1+\varepsilon}}{g+\varepsilon}A_1 + v_2A_2\right). \tag{24}$$

Here $\beta_1 = q\eta_1$; $\beta_2 = \sqrt{1+\varepsilon}\,\beta_1$; $\varepsilon = \eta_2^2/\eta_1^2 - 1$;

$$v_1 = \frac{\beta_1}{\sqrt{\tau^3 g}}\left(f_2\frac{dK_\lambda(x)}{dx} - g_2\frac{dI_\lambda(x)}{dx}\right);\quad x = \beta_1\sqrt{\frac{g}{\tau^3}};$$

$$v_2 = \frac{\beta_2}{\sqrt{\tau^3(g+\varepsilon)}}\left(-g_1\frac{dI_\lambda(y)}{dy} + f_1\frac{dK_\lambda(y)}{dy}\right);\quad y = \beta_2\sqrt{\frac{g+\varepsilon}{\tau^3}};$$

the function g being the solution of Cauchy problem (5) with $V \equiv 0$.

To construct the perturbed motion, we add the following initial conditions to the system

$$A_1(1) = A_2(1) = 0,\ N_1(1) = a_1/\eta_1,\ N_2(1) = a_2/\eta_1. \tag{25}$$

The amplitudes of the deviations of the free boundaries are determined very simply from the known functions N_1, N_2:

$$R_{n\lambda}^1 = \frac{\eta_1}{\tau m_1}N_1(\tau),\quad R_{n\lambda}^2 = \frac{\eta_1}{\tau m_2}N_2(\tau). \tag{26}$$

For two-dimensional perturbations, when $q = 0$, Eq. (20) has the form

$$A = B_1(t)(\eta^2+\tau c)^{\lambda/2} + B_2(t)(\eta^2+\tau c)^{-\lambda/2}.$$

Only the equations for $N_1(\tau)$, $N_2(\tau)$ change in this case: instead of (24), we will have the equations

$$\frac{dN_1}{d\tau} = -\frac{\lambda\tau}{\Delta}\left\{\left[\left(\frac{g}{g+\varepsilon}\right)^{\lambda/2} + \left(\frac{g+\varepsilon}{g}\right)^{\lambda/2}\right]A_1 - 2A_2\right\},$$

$$\frac{dN_2}{d\tau} = \frac{\lambda\tau}{\Delta\sqrt{1+\varepsilon}}\left\{-2A_1 + \left[\left(\frac{g}{g+\varepsilon}\right)^{\lambda/2} + \left(\frac{g+\varepsilon}{g}\right)^{\lambda/2}\right]A_2\right\},\quad \Delta = \left(\frac{g}{g+\varepsilon}\right)^{\lambda/2} - \left(\frac{g+\varepsilon}{g}\right)^{\lambda/2}. \tag{27}$$

Of course, we should put $q = 0$ ($\beta_1 = 0$) in Eqs. (23).

Finally, for purely radial perturbations ($\lambda = 0$, $q = 0$), the solution of Eq. (20) is $A = B_1(t)\ln(\eta^2 + \tau c) + B_2(t)$ and for N_1, N_2 we have

$$\frac{dN_1}{d\tau} = \frac{2\tau}{\ln(1+\varepsilon/g)}(A_1 - A_2),\quad \frac{dN_2}{d\tau} = -\frac{2\tau}{\sqrt{1+\varepsilon}\ln(1+\varepsilon/g)}(A_1 - A_2). \tag{28}$$

Here, in Eqs. (23) it is necessary to put $\lambda = 0$, $q = 0$.

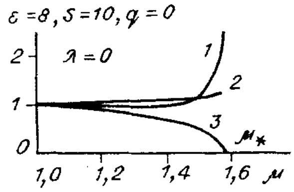

Fig. 4

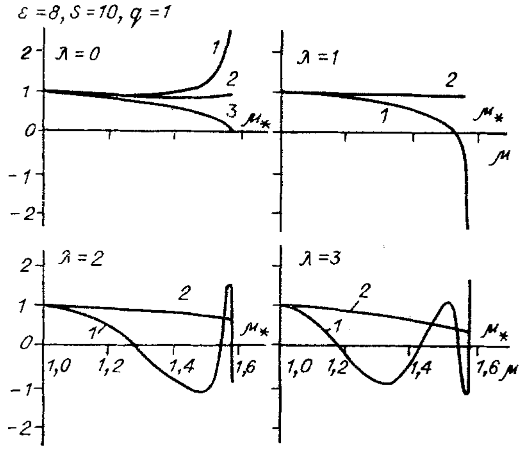

Fig. 5

The coefficients of system (23)-(24) are regular everywhere except for the point $\tau = \tau_* = 1 + kt_*$, at which $g(\tau_*) = 0$. There are logarithmic singularities at this point. For example, as $g \to 0$

$$\frac{m_{1\tau\tau}}{\tau m_1} \sim \frac{3\varepsilon\tau}{4g^2 \ln g}, \quad \frac{m_{2\tau\tau}}{\tau m_2} \sim \frac{3\varepsilon}{4\sqrt{1+\varepsilon g \ln^2 g}}.$$

The presence of such singularities makes it difficult to find the asymptotic expansion of the solution of problem (23)-(24) in the neighborhood of the singular point $\tau = \tau_*$. Nevertheless, such an expansion can be obtained by means of different substitutions. This analysis requires a great deal of calculation, and only its results are presented here.

Let $S_1 = S_2 = 0$ represent inertial collapse of the layer. It turns out that there are different cases $\lambda > 1$, $\lambda = 1$, $\lambda = 0$. Omitting the intermediate calculations, we present the asymptotes of the amplitudes of the functions R^1, R^2 at $r_1(t) \to 0$, when $t \to t_*$. Specifically,

$$R^1_{n\lambda} \sim D_1(\tau)\left[-\ln\left(\frac{r_1\sqrt{\tau}}{\eta_1}\right)\right]^{1/4} \exp\left[i\sqrt{\lambda-1}\ln\left(\frac{r_1}{\eta_1}\right)\right], \quad \lambda > 1; \tag{29}$$

$$R^1_{n1} \sim D_2(\tau)\left[-\ln\left(\frac{r_1\sqrt{\tau}}{\eta_1}\right)\right]^{1/2} \exp\left\{i\left[-2\ln\left(\frac{r_1\sqrt{\tau}}{\eta_1}\right)\right]^{1/2}\right\}, \quad \lambda = 1; \tag{30}$$

$$R^1_{n0} \sim D_3(\tau)\left[-\ln\left(\frac{r_1\sqrt{\tau}}{\eta_1}\right)\right]^{1/4}\left(\frac{r_1}{\eta_1}\right)^{-1/2}, \quad \lambda = 0 \tag{31}$$

with certain functions $D_1(\tau)$, $D_2(\tau)$, $D_3(\tau)$ bounded at $\tau \to \tau_*$. Thus, the internal surface is always unstable in the case of collapse. As regards the external surface, for all $\lambda \geqslant 0$

$$R^2_{n\lambda} \sim D_4(\tau) r_1\left[-\ln\left(\frac{r_1\sqrt{\tau}}{\eta_1}\right)\right]^{3/2}. \tag{32}$$

Thus, this surface is stable in the case of collapse. It should be noted that at $\lambda > 1$, $R^1_{n\lambda}$ behaves as in the case of the compression of a ring of fluid [5]. The given asymptote is also independent of whether the initial perturbations are related to potential or curl. When $\lambda = 1$, three-dimensional perturbations somewhat alleviate the instability but do not eliminate it completely (for the ring, $R^1_1 \sim D_2(\tau)[-\ln(r_1/\eta_1)]^{3/2}$ [5]).

Effect of Capillarity

Let $S_1 \neq 0$, $S_2 \neq 0$ in systems (5), (23). It can be shown that in this case the principal terms of the asymptote for $R^1_{n\lambda}$, $R^2_{n\lambda}$ at any fixed λ, q coincide with (29)-(32). Thus, the internal surface is unstable during collapse, but for sufficiently high harmonics with $\lambda \gg \left[-\frac{r_1}{\eta_1}\ln\left(\frac{r_1\sqrt{\tau}}{\eta_1}\right)\right]^{-1/2}$ or $q \gg \left[-\left(\frac{r_1}{\eta_1}\right)^3 \ln\left(\frac{r_1\sqrt{\tau}}{\eta_1}\right)\right]^{-1/2}$ at $r_1 \to 0$ capillary forces restrict the growth of the perturbations: $|R^1_{n\lambda}| < \infty$, $t \to t_*$.

Note 2. When $q = 0$ (plane perturbations) or $\lambda = 0$, $q = 0$ (radial perturbations), the asymptotes coincide with (29)-(32).

Figures 3-5 show the results of calculations of the amplitudes of perturbations of the boundaries of a cylindrical layer. In all of the figures, curve 1 corresponds to $R^1_{n\lambda}$, while curve 2 corresponds to $R^2_{n\lambda}$. Line 3 describes the behavior of the radius of the internal cavity r_1, $\mu_* = (1 + kt_*)^2$, $\varepsilon = 8$, $S_1 = S_2 = 10$. Figure 3 shows the behavior of two-dimensional perturbations (23), (27) with $\lambda = 1, 2, 3$, while Fig. 4 shows the characteristic curves for radial perturbations (23), (28). Figure 5 illustrates the behavior of perturbations at the free surfaces of the layer with $q = 1$, $\lambda = 0, 1, 2, 3$. The numerical curves that were constructed confirm the asymptotic results of the previous section — the internal surface is unstable during collapse, while the perturbations on the external surface die out.

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EFFECT OF THE CHOICE OF CREEP INSTABILITY CRITERION ON THE SOLUTION OF THE PROBLEM OF OPTIMIZING ROD-SHAPED STRUCTURES

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There are several approaches to evaluating the stability of a structure under creep conditions [1]. Uncertainty in the selection of a criterion of instability is an obstacle to the exact formulation of the problem of optimizing rheological systems. None of the existing solutions [2, 3] combine the results of solution of the problem for different approaches. Such a combination is lacking despite the fact that these approaches differ significantly in regard to their value for predicting the critical time.

The goal of the present study is to evaluate the effect of the choice of instability criterion on the solution of the optimization problem. We will examine so-called conditional criteria [4]. We present the equations of the problem of the maximum of the critical time for an arbitrary rod-shaped structure, and we use a specific example to determine the condition of the minimum of volume for a fixed critical time. It is shown that the choice of criterion has no effect on the optimum form of the system in the first case and that the effect is negligible in the second case.

We will assume that the material of the rod obeys the creep law [5]

$$\dot{p}p^{\alpha} = f(\sigma) \tag{1}$$

($p = \varepsilon - \sigma/E$ is the creep strain; α is the strain-hardening parameter). Analyzing variants of conditional instability criteria for creep, we note that for most of them the critical strain for a compressed rod can be represented in the form

$$p = \varphi(\sigma_0 - \sigma)/E, \tag{2}$$

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